Tactical Decompositions and Orbits of Projective Groups

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ABSTRACT

We consider decompositions of the incidence structure of points and lines of $PG(n, q)$ $(n \geq 3)$ with equally many point and line classes. Such a decomposition, if line-tactical, must also be point-tactical. (This holds more generally in any 2-design.) We conjecture that such a tactical decomposition with more than one class has either a singleton point class, or just two point classes, one of which is a hyperplane. Using the previously mentioned result, we reduce the conjecture to the case $n = 3$, and prove it when $q^2 + q + 1$ is prime and for very small values of q. The truth of the conjecture would imply that an irreducible collineation group of $PG(n, q)$ $(n \geq 3)$ with equally many point and line orbits is line-transitive (and hence known).

1. INTRODUCTION

It is well known that a collineation group of a finite projective space $PG(n, q)$ has at least as many orbits on lines as on points. This paper reports on an attempt to determine which collineation groups have equally many point orbits and line orbits. For $n = 2$, any group has this property, and the problem is simply the determination of all subgroups of $P\Gamma L(3, q)$; we ignore this case. However, for $n > 2$, the position is very different. We conjecture that such a group is line-transitive, or fixes a hyperplane and acts line-transitively on it, or (dually) fixes a point and acts line-transitively on the quotient space. (Note that all line-transitive collineation groups have been determined:

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see [1], [3].) We prove the conjecture for certain values of q, including all $q<9$ (though detailed calculations for $q=7$ are not given), and for reducible groups. The case $q=2$ has been independently handled by Saxl [4].

The combinatorial analogue of our situation is a tactical decomposition of the point-line design of $PG(n,q)$ with equally many point and line classes. We show that such a decomposition induces a decomposition with the same property of any 3-dimensional subspace, even when nothing is known about the group induced on such a subspace. Thus we need consider in detail only the case $n=3$. In fact, some of our results have greater generality.

In Section 2 we prove that a block-tactical decomposition of a 2-design with equally many point and block classes is necessarily point-tactical. We also obtain numerical information about such a decomposition. In Section 3, we consider certain "special" classes of lines in $PG(3,q)$, which can be defined by a number of properties, including the condition that such a class contains a fixed number of lines of any spread. A line class in a symmetric tactical decomposition is special, but not conversely. Combining the approaches, we are able to describe fairly precisely the symmetric tactical decompositions of $PG(3,q)$ for some q; this is done in Section 4, and the consequences for our problem on collineation groups are drawn in Section 5. The problems of determining the symmetric tactical decompositions and the special line classes in $PG(3,q)$ can be regarded as successive combinatorial generalizations of our group-theoretic problem. We remark that even if the original problem were solved by other methods, these geometric questions would still be of interest.

2. SYMMETRIC TACTICAL DECOMPOSITIONS

Let $\mathcal{G}$ be an incidence structure of points and blocks. Suppose the points are partitioned into nonempty classes $\mathcal{P}_1,\ldots,\mathcal{P}_s$, and the blocks into nonempty classes $\mathcal{B}_1,\ldots,\mathcal{B}_t$. This decomposition is called *block-tactical* if there is an $s\times t$ matrix $A=(a_{ij})$ such that $|\mathcal{P}_i\cap B|=a_{ij}$ for any $B\in\mathcal{B}_j$; *point-tactical* if there is an $s\times t$ matrix $B=(b_{ij})$ such that $|(p)\cap\mathcal{B}_j|=b_{ij}$ for any $p\in\mathcal{P}_i$; and *tactical* if it is both point- and block-tactical. [We identify a block with the set of points incident with it; and (p) denotes the set of blocks incident with p.]

The following result is a well-known consequence of Block's lemma (see [2, p. 21]):

Proposition 2.1. *Suppose the incidence matrix of $\mathcal{G}$ has rank equal to the number of points. Then, for any block-tactical decomposition, the matrix A has rank s, and in particular $t\geqslant s$.*

An important class of incidence structures satisfying the hypothesis of Proposition 2.1 is the class of 2-designs: see the standard proof of Fisher's inequality [2, p. 20]. For these, there is a remarkable consequence of equality in the above inequality. We use the usual parameters r, k, λ for 2-designs.

PROPOSITION 2.2. *Suppose $\mathcal{I}$ is a 2-design. Then a block-tactical decomposition with equally many point and block classes is tactical.*

Proof. Let V_1 (V_2) denote the vector space of functions from the point set (block set) to $\mathbb{Q}$. Let $f_1,\ldots,f_s$ be the characteristic functions of the point classes (in V_1), and $g_1,\ldots,g_s$ those of the block classes (in V_2). The incidence matrix of the design represents the linear transformation $\alpha : V_1 \to V_2$ defined by the rule

$$(f\alpha)(B) = \sum_{p \in B} f(p).$$

(Our earlier remarks imply that α is a monomorphism.)

From the definition of a block-tactical decomposition, we have

$$f_j\alpha = \sum_{i=1}^{s} g_i a_{ij}.$$

Now $\alpha\alpha^t = (r-\lambda)I + \lambda J$, where I and J are represented by the identity and all-1 matrices. So

$$\begin{aligned} f_j\alpha\alpha^t &= f_j[(r-\lambda)I + \lambda J] \\ &= (r-\lambda)f_j + \lambda v_j e, \end{aligned}$$

where e denotes the constant function with value 1, and v_j is the cardinality of $\mathcal{P}_j$. By Proposition 2.1, A is nonsingular; let $C = (c_{ij})$ be its inverse. Then

$$\begin{aligned} \sum_j c_{jh}(f_j\alpha)\alpha^t &= \sum_{i,j} (g_i\alpha^t)a_{ij}c_{jh} \\ &= g_h\alpha^t. \end{aligned}$$

Thus $g_h\alpha^t$, whose value at p is the number of blocks of the hth class containing p, is a known linear combination of $f_1,\ldots,f_s$; precisely, $g_h\alpha^t = \Sigma_j b_{hj}f_j$, where $b_{ij} = (r-\lambda)c_{ji} + \lambda\Sigma_k v_k c_{ki}$. Thus, the decomposition is point-tactical. ∎

We call a tactical decomposition of a 2-design *symmetric* if it has equally many point and block classes.

REMARK. Proposition 2.1 can be proved by observing that α is a monomorphism and A is a matrix representing its restriction to the subspace $\langle f_1,\ldots,f_s\rangle$.

Let V be the diagonal matrix $(v_i\delta_{ij})$, and W the matrix $(w_i\delta_{ij})$, where $w_i = |\mathcal{B}_i|$. Easy counting arguments show that $AB^t = (r-\lambda)I + \lambda VJ$, $JA = kJ$, $BJ = rJ$, and $VB = AW$. Then

$$\begin{aligned} AB^tA &= (r-\lambda)A + \lambda VJA \\ &= (r-\lambda)A + \lambda kVJ \\ &= (r-\lambda)A + (\lambda k/r)VBJ \\ &= (r-\lambda)A + (\lambda k/r)AWJ. \end{aligned}$$

Since A is nonsingular,

$$B^tA = (r-\lambda)I + (\lambda k/r)WJ. \tag{$*$}$$

This equation contains nontrivial information. For example:

COROLLARY 2.3. *If $(r,\lambda k)=1$, then each block class has size divisible by r; and if in addition r is prime, then there is a point class meeting every block in 0 or c points, for some constant c.*

Proof. The first assertion is clear from $(*)$. If r is prime, then $r\nmid v$, so some point class has size v_i coprime to r. From the equation $VB = AW$, we see that b_{ij} is divisible by r for all j. But a point of $\mathcal{P}_i$ lies on only r blocks. So all blocks meeting $\mathcal{P}_i$ lie in the same class, and the result follows. ■

COROLLARY 2.4. *If $\lambda = 1$, then the number of blocks of $\mathcal{B}_j$ meeting a given block of $\mathcal{B}_i$ is $(\lambda k/r)w_j + (r-k-\lambda)\delta_{ij}$.*

Proof. Immediate from (*), since such blocks meet in just one point. ■

COROLLARY 2.5. *$rk(r-\lambda)^{s-1}\prod v_i/\prod w_i$ and $rk(r-\lambda)^{s-1}\prod w_i/\prod v_i$ are both integer squares.*

Proof. $\det A \cdot \det B = rk(r-\lambda)^{s-1}$, and $\det B/\det A = \prod w_i / \prod v_i$. So the given expressions are $(\det A)^2$ and $(\det B)^2$. ■

REMARK. Proposition 2.2 is not valid for the wider class of incidence structures satisfying the hypotheses of Proposition 2.1. For example, let $\mathcal{G}$ be the structure whose points and blocks are the 2-subsets and 3-subsets of $\{1,\ldots,6\}$, with incidence = inclusion. Partition the points of into $\{\{1,2\}, \{3,4\}\}$ and its complement. We may regard the members of the first class as red edges of the complete graph K_6, those of the second class as blue edges. Any 3-set contains either one red edge or none at all; the resulting partition of the blocks gives rise to a decomposition which is obviously block-tactical, with $s=t=2$. However, it is not point-tactical: the edge $\{1,3\}$ lies in two triples containing red edges, $\{1,5\}$ lies in one such triple, and $\{5,6\}$ in none at all. We do not know if there is any class of structures more general than 2-designs for which Proposition 2.2 is valid.

3. SPECIAL LINE CLASSES IN PG(3, q)

In this section, we take $\mathcal{G}$ to be the 2-design of points and lines in PG(3, q), with $r = q^2+q+1$, $k=q+1$, $\lambda=1$. We use the notation of Section 2.

PROPOSITION 3.1. *Let $\mathcal{L}$ be a set of lines of* PG(3, q) *with characteristic function* g. *Then the following assertions about $\mathcal{L}$ are equivalent*:

(i) *g lies in the range $R(\alpha)$ of α*;

(ii) *g is orthogonal to the kernel $K(\alpha^t)$ of α^t*;

(iii) *for any regulus R, $|\mathcal{L}\cap R| = |\mathcal{L}\cap R^{\mathrm{opp}}|$, where R^{opp} denotes the opposite regulus*;

(iv) *there is a number x such that $|\mathcal{L}\cap \mathrm{S}| = x$ for any spread* S;

(v) *there is a number x such that $|\mathcal{L}\cap \mathrm{S}| = x$ for any regular spread* S.

Proof. We have $V_2 = R(\alpha) \perp K(\alpha^t)$, so (i) and (ii) are equivalent. This decomposition is invariant under the group $G = \mathrm{PGL}(4,q)$ of projectivities; and $R(\alpha)$ splits into two G-invariant subspaces, namely $\langle e\rangle$ and $\langle e\rangle^{\perp}\cap R(\alpha)$, where e is the constant function with value 1. Since G has rank 3 on the set of lines of PG(3, q), V_2 is the sum of three irreducible G-modules; so G acts irreducibly on $K(\alpha^t)$. Thus $K(\alpha^t)$ is spanned by any of its nonzero functions together with its images under G. If $\chi(T)$ denotes the characteristic function of the set T of lines, the equivalence of (ii) with (iii)–(v) is now shown by using the functions $\chi(R)-\chi(R^{\mathrm{opp}})$ and $\chi(\mathrm{S})-\chi(\mathrm{S}')$ in $K(\alpha^t)$, where S and

S' are spreads (or else regular spreads). Of course, by choosing other functions in $K(\alpha^t)$, we could easily obtain further equivalent conditions. ■

We call a class of lines satisfying the assertions of Proposition 3.1 *special*; the number x in (iv) and (v) is called its *parameter*.

PROPOSITION 3.2. *Any line class in a symmetric tactical decomposition of* PG$(3, q)$ *is special.*

Proof. If $f_1, \ldots, f_s$ are the characteristic functions of the point classes and $g_1, \ldots, g_s$ those of the line classes, we have $f_j\alpha = \Sigma g_i a_{ij}$, where $A = (a_{ij})$ is nonsingular; so $g_i \in \langle f_1\alpha, \ldots, f_s\alpha \rangle \subseteq R(\alpha)$ for $1 \leqslant i \leqslant s$.

PROPOSITION 3.3. *Let* $\mathfrak{L}$ *be a special line class with parameter* x *and characteristic function* g. *Then*

(i) $|\mathfrak{L}| = x(q^2 + q + 1)$;

(ii) *for any line* L, *the number of members of* $\mathfrak{L}$ *skew to* L *is* $[x - g(L)]q^2$;

(iii) *for any two skew lines* L_1, L_2, *the number of members of* $\mathfrak{L}$ *skew to* L_1 *and* L_2 *is* $[x - g(L_1) - g(L_2)]q(q-1)$.

Proof. The group $G = \mathrm{PGL}(4, q)$ operates transitively on triples of pairwise skew lines. Hence, for $i \leqslant 3$, the number n_i of spreads containing i pairwise skew lines depends only on i; and by counting we find that $n_1 = n_0/(q^2 + q + 1)$, $n_2 = n_1/q^2$, $n_3 = n_2/q(q-1)$. To prove (i), count pairs (L, S), where L is a line of L, and S a spread containing L, to find that $|\mathfrak{L}|n_1 = n_0 x$, or $|\mathfrak{L}| = x(q^2 + q + 1)$. To prove (ii), count pairs (L', S), where L' is a line of $\mathfrak{L}$ skew to L, and S a spread containing L and L'. A similar argument gives (iii). ■

PROPOSITION 3.4.

(i) *A special line class with parameter* 1 *consists either of all lines in a plane* Π *or of all lines through a point* p.

(ii) *A special line class with parameter* 2 *consists of all lines in a plane* Π *or through a point* p, *where* $p \notin \Pi$.

Proof. (i): According to Proposition 3.3(ii), if $x = 1$ then no two lines of $\mathfrak{L}$ are disjoint. Any maximal set of pairwise intersecting lines is of one of the types given in (i), and has cardinality $q^2 + q + 1$. By Proposition 3.3(i), $\mathfrak{L}$ itself is maximal.

(ii): Let $\mathfrak{L}$ be a special line class with parameter 2. Select skew lines $L_1, L_2 \in \mathfrak{L}$. Then $2(q+1)$ lines of $\mathfrak{L}$ meet both L_1 and L_2, and no three of these are skew. This can only occur if all such lines meet one of L_1 and L_2 (say L_1) in either of two points, and lie in either of two planes on the other.

We say that L_1 is of point type and L_2 of plane type. To justify the terminology, we must show that L_1 is of point type in any skew pair in which it occurs, and dually. So suppose that L_2' is skew to L_1. Then $L_2 \cap L_2'$ is a point, p say, and p is joined to exactly two points of L_1; so L_2' cannot be of point type.

Thus the lines of $\mathfrak{L}$ are partitioned into two classes, in such a way that skew lines belong to different classes. So two lines of the same class meet, whence each class is either concurrent or coplanar. The conclusion follows. ■

We conjecture that the only special classes of lines are the empty class: those of Proposition 3.4, and the complements of these (with parameters $0, 1, 2, q^2-1, q^2, q^2+1$). It follows from Proposition 3.4 that the conjecture is true for $q = 2$.

Remark. A special class with parameter 2 cannot occur as a line class in a symmetric tactical decomposition. For p is the unique point lying only on lines of the class $\mathfrak{L}$, so it must form a singleton point class; but no line of $\mathfrak{L}$ in Π contains p.

4. TACTICAL DECOMPOSITIONS OF PG(n, q)

Let $A(n, q)$ be the following assertion: Any symmetric tactical decomposition of the point-line design of PG(n, q) (with more than one point class) either has a singleton point class $\{p\}$ and a line class consisting of all lines on p, or has just two point classes (a hyperplane H and its complement) and two line classes (the lines in H and those meeting H in a point).

We conjecture that $A(n, q)$ holds for all q and all $n \geqslant 3$. In support of the conjecture, we offer the following pieces of evidence.

Proposition 4.1. *For any q and any $n \geqslant 3$, $A(3, q)$ implies $A(n, q)$.*

Proof. Suppose we have a tactical decomposition of PG(n, q), and let E be any 3-space. We observe that the induced decomposition of E is block-tactical. We can arrange the decomposition matrix A so that the classes represented in E come first; then

$$A = \begin{pmatrix} A_1 & C \\ 0 & D \end{pmatrix}$$

(the 0 because if a line class $\mathfrak{L}$ is represented in E, then so are all the point classes which meet lines in $\mathfrak{L}$).

Suppose A has size $s_0 \times t_0$. Since the decomposition of E is block-tactical, we have $t_0 \geq s_0$ (by Proposition 2.1); and since A is nonsingular (again by Proposition 2.1) we have $s_0 \geq t_0$. So $s_0 = t_0$, and the decomposition of E is tactical and symmetric (by Proposition 2.2).

Now suppose $A(3, q)$ holds. Then a 3-space E is of one of three types:

(i) all points of E lie in the same class, as do all lines of E;
(ii) E meets some point class in a single point p;
(iii) E meets just two point classes, one of them in a plane Π.

Suppose there is a 3-space E of type (ii), with distinguished point p. Let $\mathfrak{L}$ be the line class containing all lines on p (and no other line of E). For any 3-space E' such that $E \cap E'$ is a plane containing p, all lines on p in $E \cap E'$ (and no other lines of $E \cap E'$) belong to $\mathfrak{L}$; by $A(3, q)$, E' is of type (ii), and all its lines on p belong to $\mathfrak{L}$. Thus every line on p in the whole space, and no other line, belongs to $\mathfrak{L}$.

Next, suppose there is a 3-space E of type (iii), but none of type (ii). Let $\mathfrak{P}$ be the point class meeting E in Π. Then $\mathfrak{P}$ meets every 3-space in the empty set, a plane, the complement of a plane, or the whole space. But, if $E \cap E'$ is a plane, then $\mathfrak{P} \cap E'$ contains at least a line, and so cannot be empty or the complement of a plane. By connectedness, $\mathfrak{P}$ meets every 3-space in either the whole space or a plane. So $\mathfrak{P}$ is a subspace, necessarily a hyperplane.

Finally, if every 3-space meets only one point class, then there is only one point class. ∎

PROPOSITION 4.2. *$A(3, q)$ holds if $q^2 + q + 1$ is prime.*

Proof. Take a tactical decomposition of PG$(3, q)$. By Corollary 2.3, there is a point class $\mathfrak{P}$ and a number h such that $|\mathfrak{P} \cap L| = 0$ or $h + 1$ for any line L. If $h = 0$, then $\mathfrak{P}$ is a singleton; so suppose $h > 0$. Also, we may suppose $h < q$. If W is an i-space, then $|\mathfrak{P} \cap W| = 0$ or $1 + [(q^i - 1)/(q - 1)]h$; in particular, $|\mathfrak{P}| = 1 + (q^2 + q + 1)h$.

Take a line L disjoint from $\mathcal{P}$. The planes through L partition into sets of size $1+(q+1)h$; so $1+(q+1)h$ divides $1+(q^2+q+1)h$, whence $1+(q+1)h$ divides q^2. Since $q^2 \equiv 1 \pmod{q+1}$, we have $[1+(q+1)h][1+(q+1)g] = q^2$ for some integer g; and $h>0$ forces $g=0$, $h=q-1$. Then the complement of $\mathcal{P}$ meets every line in 1 or $q+1$ points, and so is a plane Π; and every point of Π lies on a line meeting $\mathcal{P}$, so Π is a single point class. ■

This verifies $A(3,q)$ for 11 of the 35 prime powers below 100. The next two lemmas aid the study of special values of q.

LEMMA 1. *If A is the matrix of a symmetric tactical decomposition of* $\mathrm{PG}(n,q)$ *($n \geqslant 2$, $q=p^a$, p prime), then* $|\det A| = (q+1)p^b$ *for some b.*

Proof. Consider first the case $n=2$. Then, as in Section 2, $\det A \cdot \det B = rk(r-\lambda)^{s-1} = (q+1)^2 q^{s-1}$. Since all column sums of A and all row sums of B are $q+1$, we have $q+1 \mid \det A$, $q+1 \mid \det B$, and the result follows.

Now suppose $n>2$. Our proof is by induction on the number s of classes. The result is clear for $s=1$; so suppose $s>1$. If there is a plane meeting every point class, then the result follows from the preceding paragraph; so suppose not. Certainly there is a plane Π meeting more than one point class. Order the classes so that those represented in Π come first. As in the proof of Proposition 4.1, we have

$$A = \begin{pmatrix} A_1 & C \\ 0 & A_2 \end{pmatrix},$$

so $\det A = \det A_1 \cdot \det A_2$. Now $|\det A_1|$ is $q+1$ times a power of p, by the preceding paragraph. If we amalgamate all the point classes and all the line classes represented within Π, we obtain a symmetric tactical decomposition with fewer classes; its matrix is

$$\begin{pmatrix} q+1 & c \\ 0 & A_2 \end{pmatrix},$$

where c is the vector of column sums of C. By the inductive hypothesis, $|\det A_2|$ is a power of p. This proves the lemma. ■

LEMMA 2. *Let $q^2+q+1 = p_1^{a_1}\cdots p_t^{a_t}$, where $p_1,\ldots,p_t$ are distinct primes and $a_1,\ldots,a_t > 0$. Then, in any symmetric tactical decomposition of* $\mathrm{PG}(3,q)$, *there is just one point class whose cardinality v_j is coprime to p_i (for each i); we have $v_j \equiv 1 \pmod{p_i^{a_i}}$ and $v_k \equiv 0 \pmod{p_i^{a_i}}$ for $k \neq j$.*

Proof. We know that q^2+q+1 divides the cardinality of each line class. Also, $\Sigma v_i = (q+1)(q^2+1)$, so there is an index j for which $p_i \nmid v_j$. From the matrix equation $VB = AW$, we see that $p_i^{a_i}$ divides every entry of the jth row of B. But $\det A \cdot \det B = (q^2+q+1)q^{s-1}(q+1)^s$; so no further row of B can be divisible by p_i. Thus $p_i^{a_i} \mid v_k$ for $k \neq j$, whence $v_j \equiv 1 \pmod{p_i^{a_i}}$. ■

PROPOSITION 4.3. *$A(3,4)$ is true.*

Proof. By the argument of Proposition 4.2, we know that the assertion holds if there is a point class whose size is coprime to 21; so suppose not. By Lemma 2, there is a class $\mathcal{P}_1$ of size $v_1 \equiv 1 \pmod 7$, and a class $\mathcal{P}_2$ of size $v_2 \equiv 1 \pmod 3$. Thus we have $v_1 \equiv 15 \pmod{21}$, $v_2 \equiv 7 \pmod{21}$, and $21 \mid v_i$ for $i \geqslant 3$. Also, $w_i = 21x_i$ for each i, where $x_i \geqslant 3$ (by Proposition 3.4) and $\Sigma x_i = 17$.

We cannot have $v_2 = 7$, since then at most 21 lines would meet $\mathcal{P}_2$ more than once.

Suppose $v_1 = 15$. No line meets $\mathcal{P}_1$ in three or more points, since there could be at most 35 such lines. So 105 lines (secants) meet $\mathcal{P}_1$ in two points, and 105 lines (tangents) meet it in one point; each set is a single line class. We now have $(x_i) = (5,5,7)$ or $(5,5,3,4)$, while $(v_i) = (15,28,42)$, $(15,49,21)$ or $(15,28,21,21)$. Now, since

$$5 \times 21 \times \frac{20^{s-1}\prod v_i}{21^s \prod x_i} = (5 \times 2^b)^2, \qquad (**)$$

the only possibility is $s = 3$, $(v_i) = (15,49,21)$. Now $\mathcal{P}_2$ cannot meet any secant or tangent, since the equation $VB = AB$ shows that $7 \mid x_i$ for any line class whose lines meet $\mathcal{P}_2$. Thus each point of $\mathcal{P}_3$ lies on $105 \times 3/21 = 15$ secants and $105 \times 4/21 = 20$ tangents, which is clearly impossible.

For $v_1 > 15$, the possibilities for (v_i) are $(36,28,21)$, $(36,49)$, or $(57,28)$. All are easily eliminated using $(**)$. ■

Using similar arguments we have verified $A(3,7)$. We do not give details here. In particular, $A(3,q)$ holds for all $q < 9$.

5. ORBITS OF COLLINEATION GROUPS

We consider collineation groups of PG(n, q), $n \geqslant 3$, having equally many point orbits and line orbits.

PROPOSITION 5.1. *Let G be a reducible collineation group of* PG(n, q), $n \geqslant 3$, *with equally many point and line orbits. Then either*

(i) *G fixes a hyperplane H and is transitive on the lines of H; or*

(ii) *G fixes a point p and is transitive on the planes through p (or possibly both).*

Proof. We observe first that if G is any group with the stated property, then any supergroup G_1 also has this property. For, in the notation of Section 2, the condition is equivalent to the assertion that any G-invariant function in V_2 lies in $R(\alpha)$, and any G_1-invariant function is of course G-invariant.

Suppose first that G fixes an r-space W with $0 < r < n-1$. Then the supergroup G_1 consisting of all collineations fixing W has two point orbits (W and its complement), and three line orbits (lines in W, those meeting W in a point, and those disjoint from W), a contradiction.

Suppose next that G fixes a hyperplane. Let G_1 be the supergroup of G obtained by adjoining all elations with axis H. (Note that G_1 has the same action on H as G does.) Let G have r_1 point orbits and r_2 line orbits in H. Then $r_2 \geqslant r_1$. Now G_1 has $1 + r_1$ point orbits and $r_1 + r_2$ line orbits on the whole space, since the elation group acts transitively on the complement of H and on the set of lines meeting H in any given point. By the first paragraph, $1 + r_1 = r_1 + r_2$, whence $r_1 = r_2 = 1$.

The case when G fixes a point is dual. ■

PROPOSITION 5.2. *Let G be an irreducible collineation group of* PG(n, q), $n \geqslant 3$, *with equally many point and line orbits. If* A$(3, q)$ *is true, then G is line-transitive.*

Proof. The orbits of G form a symmetric tactical decomposition. ■

REMARK. The case $q = 2$ of Propositions 5.1 and 5.2 has been obtained independently by Saxl [4], who considered the more general problem of Steiner triple systems admitting groups with equally many point and triple orbits.

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